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A Generalization of Entanglement to Convex Operational Theories: Entanglement Relative to a Subspace of Observables

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We define what it means for a state in a convex cone of states on a space of observables to be *generalized-entangled* relative to a subspace of the observables, in a general ordered linear spaces framework for operational theories. This extends the notion of ordinary entanglement in quantum information theory to a much more general framework. Some important special cases are described, in which the distinguished observables are subspaces of the observables of a quantum system, leading to results like the identification of generalized unentangled states with Lie-group-theoretic coherent states when the special observables form an irreducibly represented Lie algebra. Some open problems, including that of generalizing the semigroup of local operations with classical communication to the convex cones case, are discussed.

KEY WORDS: entanglement; convex cones; ordered linear spaces; coherent states; Lie algebras; Lie groups; operational theories; observables; local operations.

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1. INTRODUCTION

Entanglement is a characteristically quantum phenomenon whereby a pure state of a composite quantum system may cease to be determined by the states of its constituent subsystems (Schrödinger, 1935). Entangled pure states are those that have *mixed* subsystem states. To determine an entangled state requires knowledge of the correlations between the subsystems. As no pure state of a classical system can be correlated, such correlations are intrinsically non-classical, as strikingly manifested by the possibility to violating local realism and Bell's inequalities (Bell, 1993). In the science of quantum information processing

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(QIP), entanglement is regarded as the defining resource for quantum communication and an essential feature needed for unlocking the power of quantum computation.

The standard definition of quantum entanglement requires a preferred partition of the overall system into subsystems—that is, mathematically, a factorization of the Hilbert space as a tensor product. Even within quantum mechanics, there are motivations for going beyond such subsystem-based notion of entanglement. Whenever indistinguishable particles are sufficiently close to each other, quantum statistics forces the accessible state space to be a proper subspace of the full tensor product space, and exchange correlations arise that are not a usable resource in the conventional QIP sense. Thus, the natural identification of particles with preferred subsystems becomes problematic. Even if a distinguishable-subsystem structure may be associated to degrees of freedom different from the original particles (such as a set of position or momentum modes, as in Zanardi (2002)), inequivalent factorizations may occur on the same footing. Entanglement-like notions not tied to modes have been proposed for bosons and fermions (Eckert *et al.*, 2002). Finally, the introduction of quasiparticles, or the purposeful transformation of the algebraic language used to analyze the system (Batista and Ortiz, 2001; Batista *et al.*, 2002), may further complicate the choice of preferred subsystems.

In this paper, we describe a notion of *generalized entanglement* (GE) introduced in Barnum *et al.* (2003a), which incorporates the entanglement settings introduced to date in a unifying framework. In quantum mechanical settings, the key idea behind GE is that entanglement is an *observer-dependent concept*, whose properties are determined by the expectations of a *distinguished subspace of observables* of the system of interest, without reference to a preferred subsystem decomposition. Distinguished observables may represent a limited means of manipulating and measuring the system. Standard entanglement is recovered when these means are restricted to arbitrary *local* observables acting on subsystems. The central idea is to generalize the observation that standard entangled pure states are precisely those that look mixed to local observers.

The most fundamental aspects of this notion of GE make use only of the convex structures of the spaces of quantum states and observables, which makes it also applicable in contexts much broader than that of quantum systems with distinguished subspaces of observables. It may be formulated within general convex frameworks, based on ordered linear spaces or the closely related notion of convex effect algebras, suitable for investigating the foundations of quantum mechanics and related physical theories (cf. Beltrametti and Bugajski, 1997, and references therein). While commenting on physically motivated special cases, we will concentrate on this general setting in the present paper.

2. MATHEMATICAL BACKGROUND

Definition 2.1. A *positive cone* is a proper subset K of a real vector space V closed under multiplication by positive scalars. It is called *regular* if it is (a) convex (equivalently, closed under addition: $K + K = K$), (b) generating ($K - K = V$, equivalently K linearly generates V), (c) pointed ($K \cap -K = \{0\}$, so that it contains no non-null subspace of V), and (d) topologically closed (in the Euclidean metric topology, for finite dimension). In the remainder of this paper, "cone" will mean a regular cone in a *finite-dimensional* vector space unless otherwise stated.

A regular cone K induces a partial order $\geq_K$ on V, defined by $x \geq_K y := x - y \in K$. It is "linear-compatible": Inequalities can be added, and multiplied by positive scalars. If one removes the requirement that the cones be generating, cones are in one-to-one correspondence with linear-compatible partial orderings. A pair $\langle V, \succeq \rangle$ of a linear space and a distinguished such ordering is called an *ordered linear space*. The categories of real linear spaces with distinguished cones and partially ordered linear spaces are equivalent.

The topological closure condition guarantees, through the Krein–Mil/man theorem, that such a cone is generated by its *extreme rays*. A *ray* belonging to a cone K is a set R such that there exists an $x \in K$ for which $R = \{\lambda x : \lambda \geq 0\}$, i.e., it is a set of nonnegative scalar multiples of some element of the cone. An *extreme ray* in K is a ray R such that no $y \in R$ can be written as a convex combination of elements of K that are not in R.

Note that the intersection of a generating cone with a subspace is (if nonempty) a cone that generates the subspace. When a cone or other set is said to generate a linear space, it does so via linear combination. When a set is said to generate a cone, it does so via positive linear combination.

A linear functional $\lambda : V \mapsto \mathbb{R}$ is said to separate C from $-C$ if $\lambda(x) \geq 0$ for all nonzero $x \in C$.

The dual vector space V^* for real V is the space of linear functions ("functionals") from V to $\mathbb{R}$; the dual cone $C^* \subset V^*$ of the cone $C \subset V$ is the set of such linear functionals that are nonnegative on C. For a slight improvement in clarity below, for $\alpha \in V^*$, $x \in V$, we write the value of α on x as $\alpha[x]$, rather than $\alpha(x)$.

The adjoint $\phi^* : V_2^* \to V_1^*$ of a linear map $\phi : V_1 \to V_2$ is defined by

$$\phi^*(\alpha)[x] = \alpha[\phi(x)], \tag{1}$$

for all $\alpha \in V_2^*$, $x \in V_1$. The following proposition is easily (but instructively) verified.

Proposition 2.2. *Let C_i be a cone in V_i for $i = 1, 2$, and let $\phi(C_1) \subseteq C_2$. Then $\phi^*(C_2^*) \subseteq C_1^*$.*

We will also use the following:

Proposition 2.3. *Let C_i be a cone in V_i for $i = 1, 2$, and let $\phi(C_1) = C_2$. Then $\phi^*(C_2^*) \subseteq C_1^*$ and ϕ^* is one-to-one.*

Proof: Let $\eta_1, \eta_2 \in C_2^*.\eta_1 \neq \eta_2$ is equivalent to the existence of y in C_2 such that $\eta_1[y] \neq \eta_2[y]$. By the assumption that ϕ maps C_1 onto C_2, there is an $x \in C_1$ such that $\phi(x) = y$; thus $\eta_1[\phi(x)] \neq \eta_2[\phi(x)]$. By the definition of ϕ^*, this implies that $\phi^*(\eta_1)[x] \neq \phi^*(\eta 2)[x]$, which implies that $\phi^*(\eta_1) \neq \phi^*(\eta_2)$. □

By an *extremal* state in a convex set of states, we just mean the usual convex-set notion that a point x is extremal in a convex set S if (and only if) it cannot be written as a nontrivial convex combination $x = \lambda_1 x_1 + \lambda_2 x_2$ of points x_1, x_2 in S. (Convex combination means $\lambda_i > 0$, $\lambda_1 + \lambda_2 = 1$, and nontrivial means $x_1 \neq x_2$). We sometimes use the physics term *pure state* for an extremal point in a convex set of states, but for clarification we emphasize that when this convex set is the set of all quantum states on some Hilbert space, the term "pure state" in the present paper refers to a projector $\pi := |\psi\rangle\langle\psi|$ and not to a vector $|\psi\rangle$ in the underlying Hilbert space.

3. GENERALIZED ENTANGLEMENT

We now define generalized entanglement of states in a convex set of states given by the intersection $\hat{C}$ of an affine "normalization" plane $\{x : \lambda(x) = \alpha\}$ (for a fixed α, which we will take to be one) with a regular cone C of "unnormalized states." This GE is a relative notion: States are entangled or unentangled relative to another such state-set $\hat{D}$, and a choice of normalization-preserving map of the first state-set onto the second, which generalizes the notion of computing the reduced density matrices of a bipartite system. To fix ideas, note that the case where C is supposed to represent states on a finite-dimensional quantum system whose Hilbert space has dimension d, C is isomorphic to the set of $d \times d$ positive semidefinite matrices, whose normalized members form the convex set of density matrices for the system, while the ambient linear space V is the space of $d \times d$ Hermitian matrices.

Definition 3.1. Let V, W be finite-dimensional real linear spaces equipped with cones $C \subset V$, $D \subset W$, and distinguished linear functionals $\lambda \in C^*$, $\tilde{\lambda} \in D^*$ that separate C, D from $-C, -D$, respectively. Let $\pi : V \to W$ be a linear map that takes C onto D (that is, $\phi(C) = D$), and maps the affine plane $L_\lambda := \{x \in V : \lambda(x) = 1\}$ onto the plane $M_{\tilde{\lambda}} := \{y \in W : \tilde{\lambda}(y) = 1\}$. An element ("state") in $\hat{C} := L_\lambda \cap C$ is called *generalized unentangled (GUE)* relative to D if it is in the closure of the convex hull of the set of extreme points x of C whose images $\pi(x)$ are extreme in D.

For convenience, we will call a pair of linear spaces V, W equipped with distinguished cones, and map π, satisfying the conditions in the above definition a *cone-pair.* We will also sometimes (it will be clear from the context) consider a cone-pair to include specified normalization functionals λ, $\tilde{\lambda}$ satisfying the conditions in the above definition, and call these the *traces* on their respective cones, so that the condition on π above may be called *trace-preservation*.

That is, with the usual physics terminology that extremal states are "pure" and nonextremal ones "mixed," unentangled pure states of C are those whose "reduced" states (images under π) are pure, and the notion extends to mixed states as in standard entanglement theory: Unentangled mixed states in C are those expressible as convex combinations of unentangled pure states (or limits of such combinations, though the latter is unnecessary in finite dimension).

It is easy to see that the motivating example of ordinary bipartite entanglement is a case of this definition. Here C is the cone of positive semidefinite (PSD) operators on some tensor product $A \otimes B$ of finite-dimensional Hilbert spaces, while D is the direct product of the cones of PSD operators on A and on B (intuitively, it is just the cone of all ordered pairs whose first member is a positive operator on A and whose second is one on B). π is just the map that takes an operator on $A \otimes B$ to the ordered pair of its "marginal" or "reduced" operators ("partial traces") on A and B. The same holds true for standard multipartite entanglement. So we may view condition (a) of this definition as based on extending the long-standing observation that for ordinary multipartite finite-dimensional quantum systems, a pure state is entangled if and only if at least one of its reduced density matrices is mixed.

It is perhaps more mathematically natural to define the *unnormalized* unentangled states of C relative to D, omitting all mention of $\lambda\tilde{\lambda}$ and the normalization-preservation requirement on π. That is:

Definition 3.2. With C, D, V, W, π a cone-pair as in Definition 3.1, $x \in C$ is *generalized unentangled* (relative to D, π) if either (a) x belongs to an extreme ray of C, and $\pi(x)$ belongs to an extreme ray of D, or (b) it is a positive linear combination of states satisfying (a), or a limit of such combinations.

It is easy to verify that the unnormalized GUE states are a (possibly nongenerating, but otherwise regular) cone in V. If one introduces the notion of normalization in C via a functional λ, it is also easily verified that the normalized GUE states of Definition 3.1 are precisely the intersection of this cone with the normalization plane. (It is straightforward to introduce a normalization plane, and associated functional $\tilde{\lambda}$, on W if desired, as the image of $L\lambda$ under π.)

Barnum *et al.* (2003a) and especially Barnum *et al.* (2003b), stressed applications in which the reduced state-set is obtained by selecting a distinguished subspace of the observables (Hermitian operators) on some quantum system. The reduced state-set is then the set of linear functionals (equivalently, consistent lists of expectation values for the distinguished observables) on this subspace of the space of all observables, induced by normalized quantum states. (It is worth noting that beyond the setting of standard quantum entanglement this is not in general a vacuous requirement: There can be normalized linear functionals on the reduced state-set that are *not* obtainable by restriction from a quantum state on the set of all observables. Although all normalized functionals on the distinguished observables can be extended in many ways to normalized functionals on the full set, in some cases not all can be extended to *positive* functionals). We dub this class of cone-pairs the *distinguished quantum observables* setting. We now show that even in the more general cones setting, there is a natural notion of observables and the abstraction of these examples embodied by Definition 3.1 can still be interpreted as restriction of the states to a subspace of the observables. To do this we employ a formalism of states, measurements, and observables that, in many variants, is frequently used as a touchstone of "operational" approaches to theories in the abstract. (By an "operational theory," we mean one in which a theory describes various measurements or operations one can perform on systems of the type described by the theory, and specifies a set of possible "states," each of which gives the probabilities for the outcomes of all possible measurements, when the system is in that state.)

We view V^* as a space of real-valued observables. For $x \in V^*$ and $\eta \in \hat{C}$, we interpret $x[\eta]$ as the expectation value of observable x in state η. We view V as the dual of V^* in such a way that $x[\eta] = \eta[x]$ for all $x \in V^*$, $\eta \in V$. But what guarantee do we have that these expectation values behave in a reasonable way, as observables in an operational theory should? That is, can we view the expectation value $\eta(x)$ of an observable x in a state η as the expected value of some quantity being measured? By this we mean that x is associated with a quantity that takes different values depending on the outcome of the measurement, and the state determines the expectation value by determining probabilities for the different outcomes of the measurement, such that the value $\eta(x)$ is indeed the expectation value of the outcome-dependent quantity, calculated according to the probabilities assigned to the outcomes by the state.

We will only sketch the answer to this question; more details may be found in many places (though accompanied by additional concepts and formalism), notably (Beltrametti and Bugajski, 1997). In the structure we have described, of state-space and dual observable space, we are able to find a special class of observables, the "decision effects," whose expectation value may be viewed as the probability of a measurement outcome. These "effects" are the elements of the initial interval $\mathcal{E} := [0, \lambda] \subset C^*$, i.e., the set of $x \in C^*$ satisfying $\lambda \geq_{C^*} x$. A (finite) *resolution of* λ is a set of effects $x_i \in \mathcal{E}$ such that $\sum_i x_i = \lambda$. For normalized states ω, it follows that $\omega(x_i) \geq 0$ and $\sum_i \omega(x_i) = 1$, so the values $\omega(x_i)$ may be viewed as probabilities of measurement outcomes, with a resolution of λ representing the mutually exclusive and exhaustive outcomes of some measurement. Then, it can be shown that for *any* observable $A \in V^*$, a resolution $\mathcal{R}$ of λ and an assignment of real values $v(x_i)$ to the outcomes in $\mathcal{R}$ can be found, such that for all normalized states ω, $\omega(A) = \sum_i \omega(x_i)v(x_i)$. (In general the converse does not hold, giving rise to a generalization of observables sometimes known as *stochastic observables*.)

We now show that our formalism of maps π onto cones D is equivalent to restriction to a subspace of observables. The formal version of this claim will consist of two propositions (one for each direction of implication in the equivalence).

Proposition 3.3. *Let V, V^*, C, C^*, λ be a cone-pair (as in Definition 3.1 ff.), and let W^* be a subspace of V^*, containing λ. For $\eta \in V$, define $\eta\lfloor : W^* \mapsto \mathbb{R}$ as the restriction of η to the subspace W^*, i.e., $\eta\lfloor(x) = \eta(x)$ for $x \in W^*$ and otherwise $\eta\lfloor(x)$ is undefined. Thus, $\eta\lfloor \in (W^*)^* \equiv W$. The restriction map $\lfloor : V \mapsto W$ has the properties required of π; that is, there is a regular cone D in W such that $\lfloor$ maps C onto it, and the image under $\lfloor$ of the plane $L_\lambda \equiv \{\eta \in V : \lambda(\eta) = 1\}$ is a translation of a plane separating D from $-D$.*

Remark: The restriction that the subspace W^* contain λ is hardly objectionable from an operational point of view. λ's expectation value is just the normalization constant, and is independent of which normalized state has been prepared. Therefore, it can be measured without any resources, and there is no point in claiming that omitting it could represent a physically significant restriction on the means available to observe or manipulate a system.

Proof: Define $D = \{\eta\lfloor : \eta \in C\}$, $M_\lambda = \{y \in W : \lambda(y) = 1\}$. The proof of Proposition 3.3 proceeds via the following claim. □

Claim 3.1. *D is a cone in W, and λ separates it from $-D$.*

Proof of Claim: It is easy to verify linearity of from the definition, and in finite dimensions, it is also easy to verify that linear maps from one vector space *onto* another (such as $\lfloor$) take regular cones to regular cones. For all $\mathbf{x} \in \dot{C}$, $\lambda[x] > 0$. But $\lambda[x] = x[\lambda]$ by duality, and by the definition of $\lfloor$ and the fact that $\lambda \in W^*$, $x[\lambda] = x[\lambda\lfloor] \equiv \lambda[x\lfloor] > 0$ for all $x \in \dot{C}$, i.e., (since $\lfloor$ maps $\dot{C}$ onto $\dot{D}$), $\lambda[y] > 0$ for ally $y \in D$. That is, λ separates D from $-D$.

The other direction of implication is:

Proposition 3.4. *Let $V, W, C, D, \lambda, \tilde{\lambda}, \pi$ be a cone-pair. Then, there exists an injection (one-to-one map) $\tau : W^* \mapsto V^*$, taking $\tilde{\lambda}$ to λ, such that π is the pullback along τ of the restriction map $\lfloor$ to $\tau(W^*)$.*

Trivial clarifying remark: For clarity, note that the definition of $\downharpoonright$ in Proposition 3.3 involved a subspace W^* of V^*; in our current context, we have defined W^* abstractly rather than as a subspace of V^*, so it is $\tau(W^*)$, which is isomorphic to W^* but *is* a subspace of V^*, to which we restrict states in defining $\downharpoonright$. (Of course, W^* *itself* is a subspace of V^* according to the category-theoretic definition of subspace.)

Proof: Let τ be π^*. That is, for all $x \in W^*$, $\eta \in V$,

$$\tau(x)[\eta] = x[\pi(\eta)]. \tag{2}$$

By duality, this gives

$$\eta[\tau(x)] = \pi(\eta)[x]. \tag{3}$$

Since, by Proposition 2.3, τ is an injection, this last equation determines $\pi(\eta)$ to be essentially $\eta\downharpoonright_{r(W^*)}$, as desired. The "essentially" refers to the fact that $\pi(\eta)$ is actually the pullback along τ of this restriction; the two are the same function only if one identifies W^* with its image under τ. τ, in other words, tells us how W^* can be identified with a subspace of the full space V^* of observables, in such a way that $\pi(\eta)$ becomes identified with the restriction of η to W^*. □

Proposition 3.5. *With our usual setup (i.e., a cone-pair), π has the property that for $x \in$ Extr D, the set $\pi^{-1}(x)$ is convex closed, and its extremal elements are extremal in C.*

Proof: Let $x \in \mathrm{Extr} D$, and let $y \in \pi^{-1}(x)$ not be extremal in C. We need to show that such a y is not extremal in $\pi^{-1}(x)$ either. $y \notin$ Extr C means there are $y_1, y_2 \in C$ with $y_1 \neq y_2$, $y = \lambda y_1 + (1-\lambda)y_2$. By linearity of π, $x \equiv \pi(y) = \lambda\pi(y_1) + (1-\lambda)\pi(y_2)$; since $x \in$ Extr (D), $\pi(y_1) = \pi(y_1) = x$. Hence, $y_1, y_2 \in \pi^{-1}(x)$, so $y \notin$ Extr π^{-1}(x). □

In important classes of examples, a stronger property holds:

Definition 3.6. A cone-pair C, D, π is said to have the *unique preimage property* (UPI property) if $x \in \mathrm{Extr} D$ implies that $\pi^{-1}(x)$ consists of a single element (which must therefore be extremal).

Equivalently (because of Proposition. 3.5), extremal reduced states have only extremal preimages.

Problem 3.7. *Find nontrivial necessary and/or sufficient conditions (some are given below, but others almost certainly exist) for cone-pairs C, D, π to have the UPI property.*

4. GENERALIZED ENTANGLEMENT IN SPECIAL CLASSES OF CONES

We now formally define several "settings" in which to study generalized entanglement; these are special classes of cone-pairs, physically and/or mathematically motivated.

Definition 4.1.

- *Distinguished quantum observables setting*, defined above. An equivalent formulation is the *Hermitian-closed (aka †-closed) operator subspace setting*, in which the the distinguished observable subspace is the Hermitian operators belonging to a †-closed subspace of the complex vector space of all linear operators on a quantum system.
- *Lie algebraic setting*. Here, C is the cone of positive Hermitian operators on a (finite-dimensional) Hilbert space carrying a Hermitian-closed Lie algebra $\mathfrak{g}$ (playing the role of W^*) of Hermitian operators (with Lie bracket $[X, Y] := i(XY - YX)$, and containing the identity operator) and D the cone (in $(W^*)^* =: W$) of linear functional on $\mathfrak{g}$ induced from positive Hermitian elements of C by restriction to W^*, via the map π.
- *Associative algebraic setting*. Here, the distinguished observables are the Hermitian elements of some associative subalgebra of the associative algebra of all operators on a quantum system.

Note that the Lie-algebraic and associative algebraic settings are special cases of the distinguished quantum observables case.

A distinction that can be nontrivially made within all the settings in this list is between those in which the distinguished observables act irreducibly, and those in which there is a nontrivial subspace invariant under the action of all observables.

Note that since the Lie-algebraic setting was defined to involve finite-dimensional †-closed matrix representations, the Lie algebras involved are necessarily reductive (Barnum *et al.*, 2003a), i.e., the direct product[4] of a semisimple and an Abelian part.

Proposition 4.2. *In the †-closed operator subspace setting, the distinguished subspace has a basis of Hermitian operators that is orthonormal in the trace inner product* $\langle A, B\rangle = \mathrm{tr}\, AB$.

Because of this proposition, we may construct an orthogonal projection operator (some would call it a superoperator) Π_S, acting on the space of Hermitian operators by projecting into the subspace of distinguished observables. We can

[4] As Lie algebras; the induced direct product of the algebras considered as vector spaces (i.e., without their Lie bracket structure) is also a vector space direct sum.

also use such a basis to define a measure of entanglement for pure states, the *quadratic relative purity* (although the name may be slightly misleading when the UPI property does not hold, for reasons we will explain).

Definition 4.3. Let ω be a state on a $\dagger$-closed set S of quantum observables. The quadratic purity $P(\omega)$ of a state ω is defined by letting X_α be an orthonormal (in trace inner product) basis of S. Then

$$P(\omega) := \sum_\alpha (\omega(X_\alpha))^2. \tag{4}$$

We also use variants of the purity where the common normalization constant of the orthonormal basis for S is chosen differently, for instance, so that the maximum value of $P(\omega)$ is unity.

Note that any state ω on the *full* operator space corresponds to a density operator ρ_ω, defined by the condition tr $(\rho_\omega X) = \omega(X)$ for all observables X.

Closely related to the above purity is the *quadratic relative purity* of a pure state $|\psi\rangle$ of the overall quantum system; this is defined equal to the quadratic purity of the state it induces on S, or equivalently, with X_α as above,

$$P_S(|\psi\rangle) := \sum_\alpha |\langle\psi|X_\alpha|\psi\rangle|^2. \tag{5}$$

In fact, this definition could be straightforwardly extended to mixed states ω on the full Hilbert space, as

$$P_S(\omega) := \sum_\alpha |\mathrm{tr}\, \omega X_\alpha|^2 \ . \tag{6}$$

However, a requirement for entanglement measures is convexity (Vidal, 2000) and the above extension lacks this as well as other desirable properties. We will generally extend pure-state entanglement measures μ to mixed states via the *convex roof construction*, standard in ordinary entanglement theory: The value of the measure on a mixed state ω is the infimum, over convex decompositions $\omega = \sum_i p_i \pi_i$ of ω into pure states π_i, of the average value of the pure-state measure, that is, of $\sum_i p_i \mu(\pi_i)$. This is convex by construction.

Defining Π_S as the projection superoperator onto the operator subspace S, it is easily verified that.

$$P_S(\omega) := \sum_\alpha |\mathrm{tr} \Pi_S(\rho_\omega) X_\alpha|^2 \ . \tag{7}$$

An equivalent definition is:

$$P_S(\omega) := \mathrm{tr}[\Pi_S(\rho_\omega)^2]. \tag{8}$$

For any density operator ρ, we call $\Pi_S(p)$ the associated *reduced* density operator; note that it need *not* be a positive operator on the full state space (although it is in the standard multipartite case). This is not problematic because for any PSD element R of the *distinguished* observable space, tr $\rho R \geq 0$, of course.

Proposition 4.4. *In the †-closed operator subspace setting, pure states with maximal relative quadratic purity are generalized unentangled.*

Proof: A necessary and sufficient condition for a normalized state ω on the full space to be pure is tr $(\rho_\omega^2) = 1$. (Henceforth we suppress the w-dependence of ρ.) Letting X_α be an orthonormal basis for the full space such that a subset (denoted by the letter β for the index) indexes the distinguished subspace S, with another subset (indexed by γ) indexing $S^\perp$, and $\langle X_\alpha \rangle$ for tr ρX_α. we have

$$\rho = \sum_\alpha \langle X_\alpha \rangle X_\alpha \ . \tag{9}$$

From this and orthogonality of the X_α it is easy to see that

$$\mathrm{tr}(\rho^2) = \sum_\alpha \langle X_\alpha \rangle^2 \ . \tag{10}$$

$P_S(\rho) \equiv \sum_{\beta \in S} \langle X_\beta \rangle^2$; if we normalize so that extremal overall states have tr $(\rho^2) = 1$, then clearly $P_S(\rho) = 1$ is maximal. This implies $\sum_{\gamma \in S^\perp} \langle X_\gamma \rangle^2 = 0$, which requires $\langle X_\gamma \rangle = 0$ for all $\gamma \in S^\perp$. Thus, $P_S(\rho)$ has a unique preimage, namely itself. Since w is pure this implies, immediately from the definition of the convex set of reduced states, that $P_S(\rho)$ is extremal in that set. □

Note that the converse of this statement is *not* true in general, although (as shown in the proof of the preceding proposition), it is immediate when the UPI property holds.

Proposition 4.5. *In the †-closed operator subspace setting when the UPI property holds, generalized unentangled states have maximal relative quadratic purity.*

Problem 4.6. *Is the converse of Proposition 4.5 true? That is, is it the case that when in a given instance of the †-closed operator subspace setting, every generalized unentangled state has maximal relative quadratic purity, that instance has the UPI property?*

We can also ask whether a yet stronger statement holds:

Problem 4.7. *In the †-closed operator subspace setting, does every π-image of a generalized unentangled state that has maximal relative quadratic purity have a unique preimage?*

It is not hard to see, from the representation theory of associative algebras, that the UPI property holds for the irreducible associative algebraic setting. The other case in which we know it holds is the irreducible semisimple Lie algebraic setting. In this setting, the observables consist of the Hermitian part (itself a real Lie algebra) of a complex Lie algebra represented faithfully and irreducibly by matrices acting on a finite-dimensional complex Hilbert space, and including the identity matrix I. Real semisimple algebras possibly extended by the identity are the general forms of such Hermitian parts of irreducible matrix Lie algebras. The identity is relatively unimportant since all normalized states will have the same value on it: The normalization condition will be the affine plane $\omega(I) = 1$, so the convex structure of the state space will be entirely determined by the expectation values of the traceless operators. We introduce a bit more notation in order to state a result, proved in Barnum *et al.* (2003a), that includes this and other important facts about the irreducible Lie-algebraic case.

We begin by reviewing the needed Lie representation theory (Humphreys, 1972). A *Cartan subalgebra* (CSA) $\mathfrak{c}$ of a semisimple Lie algebra $\mathfrak{h}$ is a maximal commutative subalgebra. A vector space carrying a representation of $\mathfrak{h}$ decomposes into orthogonal joint eigenspaces V_λ of the operators in $\mathfrak{c}$. That is, each V_λ consists of the set of states $|\psi\rangle$ such that for $x \in \mathfrak{c}$, $x|\psi\rangle = \lambda(x)|\psi\rangle$. The eigenvalue λ is therefore a linear functional on $\mathfrak{c}$, called the *weight* of V_λ. As an example, consider a spin-J irreducible representation of $\mathfrak{su}(2)$. Any spin component J_α (for any direction a in $\mathbb{R}^3$) spans a (one-dimensional) CSA $\mathfrak{c}_\alpha$. There are $2J + 1$ weight spaces labeled by the angular momentum along α, each spanned by a state $|M\rangle$ (for $M \in \{J, J-1, \ldots, -(J-1), -J\}$) having spin component M in direction α. Note that any two CSAs are conjugate under elements of the Lie group, manifested in the spin example by the fact that J_α transforms into any desired spin component via conjugation by a rotation in SU(2).

The subspace of operators in $\mathfrak{h}$ orthogonal in the trace inner product to $\mathfrak{c}$ can be organized into orthogonal "raising and lowering" operators, which connect different weight spaces. In the example, choosing J_z as the basis of our CSA, these are $J_\pm := (J_x \pm i\, J_y)/\sqrt{2}$. For a fixed CSA and irreducible representation, the weights generate a convex polytope; a lowest (or highest) weight is an extremal point of such a polytope, and the one-dimensional weight-spaces having those weights are known as *lowest-weight states* (in the spin example, this polytope is the interval $[J, -J]$). The set of lowest-weight states for all CSAs is the orbit of any one such state under the Lie group generated by $\mathfrak{h}$. These are the group-theoretic GCSs

(Zhang *et al.*, 1990). Notably, the GCSs attain *minimum invariant uncertainty* (Delbourgo and Fox, 1977).

So far, $\mathfrak{h}$ has been assumed to be a *real* Lie algebra of Hermitian operators. These may be thought of as a distinguished family of Hamiltonians, which generate (via $h \mapsto e^{ih}$) a Lie group of unitary operators, describing a distinguished class of reversible quantum dynamics. More generally, we might want Lie-algebraically distinguished completely positive (CP) maps, $\rho \mapsto \sum_i A_i \rho A_i^\dagger$ so as to be able to describe Lie-algebraically distinguished open-system quantum dynamics. A natural class is obtained by restricting the "Hellwig-Kraus" (HK) operators A_i to lie in the topological closure $\overline{e^{\mathfrak{h}_c \oplus \mathbb{1}}}$ of the Lie group generated by the *complex* Lie algebra $\mathfrak{h}_c \oplus \mathbb{1}$.[5] Having HK operators in a group ensures closure under composition. Using $\mathfrak{h}_c \oplus \mathbb{1}$ allows non-unitary HK operators. Topological closure introduces singular operators such as projectors. The following characterizations of GUE states (Barnum *et al.*, 2003a) demonstrate the power of the Lie-algebraic setting. In the theorem, an $\mathfrak{h}$-state is defined as a linear functional on a *complex* Lie algebra $\mathfrak{h}$ belonging to the convex set of such states induced by normalized quantum states on the full representation space. Complex-linearity ensures that the convex structure of such state space is the same as that of the states induced by taking as the distinguished observables only the Hermitian elements (a real Lie algebra we denote $\mathrm{Re}(\mathfrak{h})$), which is the definition we used above for the Lie-algebraic setting.

Theorem 4.8. *Let $\mathfrak{h}$ be a complex irreducible matrix Lie algebra, with $\mathfrak{h}_\circ$, its traceless (semisimple) part and* $\mathrm{Re}\mathfrak{h}$ *its Hermitian part.*

The following are equivalent for a density matrix ρ inducing the $\mathfrak{h}$-state λ:

(1) *λ is a pure $\mathfrak{h}$-state.*
(2) *$\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ the unique ground state of some H in* $\mathrm{Re}(\mathfrak{h})$.
(3) *$\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ a minimum-weight vector (for some simple root system of some Cartan subalgebra) of $\mathfrak{h}_\circ$.*
(4) *λ has maximum $\mathfrak{h}$-purity.*
(5) *ρ is a one-dimensional projector in* $\overline{\mathrm{e}^{\mathfrak{h}}}$.

Problem 4.9. *Does the UPI property, or the implication from GUE to maximal quadratic relative purity, hold in other natural situations?*

It is fairly easy to show by example that in the Lie-algebraic setting but without the assumption of irreducibility. the UPI property need *not* hold. A more general question suggests itself:

[5] $\mathfrak{h}_c$ is constructed by taking the complex linear span of a basis for $\mathfrak{h}$.$\mathfrak{h}_c \ominus \mathbb{1}$ guarantees inclusion of the $\mathbb{1}$.

Problem 4.10. *In the †-closed operator subspace setting, does the UPI property hold whenever the distinguished operators act irreducibly?*

5. ANALOGUES OF LOCAL MAPS

Our work on GE raises many natural questions arising from the twin (and closely related) problems of finding natural generalizations or analogues of the notions of LOCC ("local operations and classical communication") and of monotone entanglement measures (or *entanglement monotones*). The relation between the two comes from requiring that a reasonable entanglement measure be nonincreasing under LOCC operations; if one found a natural generalization of such notion of LOCC to our more general settings, it would also be natural to look for measures of GE monotone under this generalization. Here, we present some ideas and partial results, but some of the most fundamental questions remain open, so this section (in which we concentrate on generalizing LOCC; for more on GE measures, see Barnum *et al.* (2003a)) will be more open-ended than the preceding ones. Indeed, we hope that the wealth of open problems suggested in this section is taken to attest to the richness of the notion of GE, and that it will stimulate further work in the area.

The semigroup of LOCC maps (Bennett *et al.*, 1996) and the preordering it induces on states according to whether or not a given state can be transformed to another by an LOCC operation are at the core of entanglement theory. LOCC maps are precisely those implementable by using CP quantum maps on the local subsystems, and classical communication, e.g., of "measurement results," between systems. We say an *explicitly decomposed* trace-preserving map $\{M_k\}_{k\in K}$ is a set of maps M_k that sum to a trace-preserving one M. The *conditional composition* of an explicitly decomposed map $\{M_k\}_{k\in K}$ with a set of explicitly decomposed maps $N_k := \{N_{nk}\}_{n\in N_k}$ is the explicitly decomposed map $\{N_{nk} \circ M_k\}_{k\in K, n\in N_k}$. We can view each M_k as being associated with measurement outcome k, obtained (given a state ρ) with probability tr $(M_k\rho)$, and leading in that case to the state $M_k\rho M_k^\dagger$. The conditional composition of $\{M_k\}_{k\in K}$ and $\{N_{nk}\}_{n\in N_k}$ can be implemented by first applying M and then, given measurement outcome k, applying N_k. Closing the set of one-party maps (for all parties) under conditional composition gives the LOCC maps. Conditional composition can also be defined in an obvious manner for explicitly decomposed maps without the trace-preservation condition. The semigroup generated by composition of unilocal explicitly decomposed maps having a single HK operator in their decomposition, is often known as SLOCC (for *stochastic* LOCC). SLOCC represents the dynamics that are possible with local quantum measurements and classical communication conditional on a *singleset* of local measurement results, when each local measurement is performed in a manner that preserves all pure states (i.e., with a single HK operator for each outcome). The mathematical structure of SLOCC is relatively simple and tractable, as the part generated by nonsingular HK operators is the trace-noincreasing part of a representation of a product of various $\mathrm{GL}(d_i)$, with the factors acting on local systems of dimension d_i.[6]

When the distinguished observables form a semisimple Lie algebra $\mathfrak{h}$, a natural multipartite structure can be exploited to generalize LOCC. $\mathfrak{h}$ can be uniquely expressed as a direct sum of simple Lie algebras, $\mathfrak{h} = \oplus_i \mathfrak{h}_i$. A Hilbert space irreducibly representing $\mathfrak{h}$ factorizes as $\mathcal{H} = \otimes_i \mathcal{H}_i$; with $\mathfrak{h}_i$ acting nontrivially on $\mathcal{H}_i$ only. This resembles ordinary entanglement, except that the "local" systems $\mathcal{H}_i$ may not be *physically* local, and actions on them are restricted to involve operators in the topological closure of a "local" Lie group representation which need not be $\mathrm{GL}(\dim(\mathcal{H}_i))$ as in standard entanglement. For each simple algebra $\mathfrak{h}_i$ a natural restriction is to CP maps with HK operators in $\overline{e^{(\mathfrak{h}_i)_c \oplus 1\!\!1}}$. GLOCC, generalized LOCC, is the closure under conditional composition of the set of operations each of which is representable with HK operators in the topological closure of $e^{(\mathfrak{h}_i)_c \oplus 1\!\!1}$ some i.

In conventional entanglement, there is also interest in the properly larger [] set of *separable* maps (Bennett *et al.*, 2001; Dür *et al.*, 2002; Vidal, 2000), which are those representable with HK operators that are tensor products. This is a mathematically simpler set than the LOCC operations, since it is just the trace-nonincreasing part of the cone generated by SLOCC maps. A Lie-algebraic generalization of separable maps is obtained by considering the semigroup of maps whose HK operators are in $\overline{e^{\mathfrak{h}_c \oplus 1\!\!1}}$. A potential generalization of LOCC involves using spectra of operators to classify them as analogues of *single-party* operators. Yet another begins from maps that induce well-defined maps on the set of reduced states, as single-party maps do in the standard setting. These alternative proposals are discussed further in Barnum *et al.* (2003a). Here we review, with slight variations, another proposal made in that paper for how the notion of LOCC might be extended to the general convex setting. In generalizing LOCC, two aspects of LOCC must be considered: The first, that it constrains maps to have certain *locality* properties; the second, that it also constrains them to be *completely positive*. We will, to some extent, consider how these might be separately generalized to the cones setting, as well as how they may combine in generalizations of LOCC.

5.1. Attempts to Generalize Complete Positivity

A *positive* map of D is a linear map $A : V \to V$ such that $A(D) \subseteq D$. The map A is *trace preserving* if $\mathrm{tr}(x) = \mathrm{tr}(A(x))$ for all x. This definition corresponds to positive, but not necessarily CP, maps in quantum settings. Without additional

[6] We are not certain if the full LOCC semigroup is the trace-noincreasing part of the topological closure of this representations, but it seems a reasonable possibility.

algebraic structure, it is not possible to define a unique "tensor product" of cones, as would be required to distinguish between positive and CP maps (Namioka and Phelps, 1969; Wittstock, 1974) (cited in Wilce, 1992).

In a continuum of possible products of cones, there are two natural possibilities that are in a sense the two extremes. The first is the convex closure of the set of tensor-products of the cones' vectors, which for the case of the product of two quantum systems' unnormalized state spaces gives the separable (aka unentangled) states of the bipartite system. The second is to use the dual cone of the cone obtained by applying the first construction to the duals of the cones; in the quantum case, it gives the set of (unnormalized) states that are positive on product effects (this is isomorphic to the cone of positive but not CP operators between the state spaces, by the "Choi-Jamiolkowski" isomorphism between $V \otimes V$ and $\mathcal{L}(V)$). It is not clear how to pick out a natural case between these extremes in general without adding algebraic structure, except perhaps if the cones are self-dual with respect to non-degenerate inner products on the real vector spaces. In that case, one could pick a self-dual cone between the two constructions (which would give the usual state space of a bipartite system in the quantum case).

The family of positive maps of D is closed under positive combinations and hence forms a cone. In the Lie-algebraic, or even the bipartite setting, the extreme points of this cone are not easy to characterize (see, for example, Gurvits, 2002; Wilce, 1992, p. 1927). In order to generalize LOCC one might try to find abstractions of the notion of complete positivity to a more general cones setting. For in LOCC the local maps are not merely positive, but CP. One might, of course, try to recapture the idea of complete positivity by explicitly introducing a cone representing the "tensor product" extension of D and requiring extendibility or "liftability' of the map to D. Another, perhaps more uniquely determined, approach might begin from the observation that the extreme points of the cone of completely positive maps are certainly extremality preserving in the following sense: A positive map A of D is *extremality preserving* if for all extremal $x \in D$, $A(x)$ is extremal. However there are extremality preserving positive, not CP, maps. An example is partial transposition for density operators of qubits. We call a positive map that is a mixture of extremality preserving maps *q-positive*. In the bipartite setting, the family of *q-positive* maps of D is between the family of positive maps and the family of CP maps acting on density matrices on $\mathcal{H}_{ab}$. It thus might be of interest to generalize LOCC in such a way as to allow q-positive, and not merely positive, maps. Possibly most of the nice properties of LOCC maps would be shared by this broader class, and it is also possible that in conjunction with q-positivity, weaker conditions might suffice to characterize generalizations of complete positivity than would be needed without q-positivity. Also, if we strengthen q-positivity to the class of extremality-preserving positive maps with positive inverses (and limits of sequences of these), we obtain in the quantum case *decomposable* maps—the cone of maps generated by the extremal CP maps (conjugation by a single Kraus operator) and the extremal completely co-positive maps (transposition, followed by conjugation by a single Kraus operator). One might consider the semigroup generated by conditional composition of *decomposable* maps that are suitable analogues of the unilocal quantum ones. It is even possible that positivity of the overall maps might put severe restrictions on the unilocal ones, probably sufficient in the quantum case with more than one system to rule out the completely co-positive ones (if this is so, it would be because the non-complete-positivity of partial transpose is detectable merely by tensoring in a single qubit—i.e., partial transpose is not even 2-positive). However, it is not so clear why this, or q-positivity is an *operationally* (as opposed to mathematically) natural requirement.

5.2. Attempts to Generalize Locality

Another approach to GLOCC, and the one we will concentrate on here, is to hope that the restriction to CP maps might either emerge, or be imposed, late in the game, concentrating instead on generalizing locality. In this approach, one runs the risk of ending up with, say, locally *positive* but not necessarily completely positive maps, and classical communication, so one must hope that in such a case one will see how to exclude non-CP maps by additional natural requirements.

To try to generalize the notion of locality, we introduce the idea of *liftability*. We say that a positive map A on D can be lifted to C if A preserves the nullspace of π, or, equivalently, if there exists a positive map A' on C such that $\pi(A(x)) = A'(\pi(x))$. In this case, we say that A' is the lifting of A to C.

In the case of standard multipartite quantum entanglement, *unilocal* maps (ones that act nontrivially only on one factor) are obviously liftable to the cone of local observables; they have a well-defined action there. But so are tensor product maps $\mathcal{A} \otimes \mathcal{B} \otimes \ldots \mathcal{Z}$. and, in the case when some of the subsystems are of the same dimension, so are maps performing permutations among the isodimensional factors. In order to get LOCC, it would seem necessary to rule out the latter two cases, leaving the unilocal maps; then one can generate a semigroup from the unilocal maps by conditional composition of *explicitly decomposed* trace-preserving maps.

Problem 5.1. *Is the semigroup generated by* completely positive *local quantum maps and pairwise exchanges of isodimensional systems the full semigroup generated by conditional composition of liftable-to-local-observables explicitly decomposed maps? Does it these lines, give rise to the same partial ordering of states as standard "SLOCC"?*

Note that using liftability to define locality is of some help in ruling out local non-completely positive maps, since all maps must be positive on the overall cone. When no subsystem has dimension greater than the square root of the overall dimension, it is fully effective in imposing complete positivity. Thus, although it is strictly speaking not a generalization of LOCC, we might want to explore, in the cones setting, the semigroup of positive maps generated by conditional composition of maps liftable to the distinguished subcone, in the hope that it may enjoy many of the same properties of the LOCC maps (which would form a subsemigroup of it, in the quantum case).

Also using liftability, we can explicitly add more cones to try to capture the idea of complete positivity or to exclude maps like the swap. For example, to exclude the swap from the bipartite quantum case, it suffices to introduce cones included in C to represent density matrices on $\mathcal{H}_a$ and $\mathcal{H}_b$ and require liftability to both of these cones. This still allows product maps. Also, it involves an explicit introduction of multiple cones. In the standard multipartite quantum case, the high degeneracy of unilocal operators can also be used to help distinguish them in a way not so directly dependent on explicit introduction of cones to represent individual systems—and similarly one can use spectral information about HK operators to characterize ones that act on the *same* single system, thereby characterizing LOCC in terms of conditional composition of explicitly decomposed maps whose Kraus operators together satisfy certain spectral conditions. For more along these lines, see Barnum *et al.* (2003a). However, it is not clear how to abstract this to general cones. Perhaps one must give up and explicitly introduce subcones in order to do this; but perhaps there is something that can be done with the facial structure of the cones D, or of the cone of positive maps on D (or of other subcones of maps chosen as abstractions capturing aspects of complete positivity). A more in-depth investigation of dynamics generalizing LOCC thus remains as a challenging and many-faceted area for research, as does the investigation of measures of GE nonincreasing under such maps.

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